

## Effects of the network structural properties on its controllability

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In a recent paper, it has been suggested that the controllability of a diffusively coupled complex network, subject to localized feedback loops at some of its vertices, can be assessed by means of a Master Stability Function approach, where the network controllability is defined in terms of the spectral properties of an appropriate Laplacian matrix. Following that approach, a comparison study is reported here among different network topologies in terms of their controllability. The effects of heterogeneity in the degree distribution, as well as of degree correlation and community structure, are discussed. © 2007 American Institute of Physics. [DOI: 10.1063/1.2743098]

In recent years, synchronization of complex networks of coupled dynamical systems has become a research subject of increasing attention within the scientific community. This is partly motivated by the frequent observation of synchronization phenomena in a wide range of different contexts, ranging from biology to medical and social sciences. On the other hand, this is because synchronization is considered a paradigmatic example of phase transitions that under certain circumstances may occur when ensembles of dynamical systems are coupled together. Generally speaking, several different processes may lead to the onset of synchronization. Namely, it may be the result of a self-organized process occurring when several coupled dynamical systems, starting from different initial conditions, converge in the same dynamical evolution (which is in general not known *a priori*); or it can be provoked by some feedback loops driving the set of all the dynamical systems toward a desired predetermined reference evolution. In this paper, we focus on synchronization of diffusively coupled networks. Moreover, we consider that a subset of nodes are selected to be controlled. This assumption is motivated by real-world network observation, where a decentralized control action is often applied only to part of the nodes. For instance, pacemaker cells have been observed to regulate several functions in living organisms; other examples are present in human networks, where particular individuals, called *leaders*, have been observed to be capable to influence the network collective dynamics. Here, the controllability of a given complex network, i.e., its propensity to being controlled onto a given reference evolution by means of a decentralized control action, is defined as the width of the range of the coupling strength term among the oscillators, which stabilizes the reference evolution. A detailed comparison among different complex networks in terms of their controllability, characterized by different degree distributions, degree correlation properties, as well as community structure, will be reported.

### I. INTRODUCTION

The control of the complex dynamics which take place on networks of many interconnected units is an issue of primary importance in various fields of applied sciences. In a recent paper,<sup>1</sup> the problem of how a dynamical complex network of diffusively coupled systems can be controlled onto a synchronous evolution was studied by applying a local feedback action to a small portion of the network nodes.

In nature, there are many situations where the control of a very large complex network is an important functional requirement; this is the case, e.g., of some bodily functions, such as the contemporaneous beats of the heart cells,<sup>2</sup> or the synchronous behaviors of the cells of the suprachiasmatic nucleus in the brain, which sets the clock of the circadian bodily rhythms.<sup>3</sup> Other examples can be easily found in social networks where the formation of mass opinions and the emergence of collective behaviors are frequently observed. Generally speaking, this issue is particularly relevant to those situations where a given common behavior of all the network microscopic systems represents a functional requirement for the network dynamics at the macroscopic level.

It is worth noting here that sometimes, in the literature, the same phenomenon is also referred to as the entrainment of a network of dynamical systems; however in what follows, for simplicity, the common term control is used.

In Ref. 1, the problem of network controllability was studied via a Master Stability Function approach.<sup>4</sup> Under the hypothesis of all the network dynamical systems being identical, and the coupling being diffusive, a quantity was defined to assess the propensity of any given complex network (or lattice) to being controlled. In so doing, the network controllability was defined as a structural property, independent of the particular type of dynamics considered at the network nodes. Specifically, the network controllability was measured in terms of a simple matrix spectral index (to be precisely defined below).

This issue seems closely related to that of the network synchronizability, as studied, e.g., in Refs. 4–7; indeed, in both cases the object of study is the range of values of a defined control parameter that rules the stability of the

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synchronous/reference evolution. However, there is a fundamental difference. The network synchronization problem considers systems which autonomously settle onto a generic (not assigned *a priori*) synchronous evolution. In such a case, synchronization is achieved by means of a self-organized process, where the involved dynamical systems dynamically adjust their trajectories until they eventually converge onto the same evolution. Many phenomena in nature resemble this type of behavior, for example, the spontaneous emergence of synchronous behavior observed in populations of fireflies.<sup>8</sup>

On the other hand, a different phenomenon (kind of dynamics) takes place when an entity external to the network, or some of its nodes (which do not undergo the influence of the other network systems) are deputed to control the whole network onto a desired synchronous evolution. Specifically, in this case, part of the network dynamical systems drive all the other network systems toward their own dynamical evolution. Moreover, this difference is not only phenomenological. In mathematical terms, this corresponds to the set of the solutions, belonging to the synchronization manifold  $\{x_1 = x_2 = \dots = x_N\}$ , shrinking to the only admissible solution  $\{x_1 = x_2 = \dots = x_N = s(t)\}$ , where  $s(t)$  is the reference evolution chosen for the network. Furthermore there is another difference. In fact, the number of network dynamical modes that need to be ensured to be stable, varies when assessing the synchronizability/controllability of a given complex network (for more details, see Ref. 1 and Sec. II).

Examples of control are the synchronous beats of the heart cells that are regulated by the activity of the pacemaker cells situated at the sinoatrial node<sup>2</sup> and the circadian rhythms, observed in many living organisms, entrained by the light-dark cycle (for humans it has been shown that the intrinsic period of oscillations of the cells in the suprachiasmatic nucleus is different from the 24-hour cycle<sup>9</sup>).

Sometimes, when looking at real-world phenomena, the distinction between these two different behaviors is not so obvious. For instance, in Ref. 10 the phenomenon of menstrual synchronization among female roommates and close friends was reported and it was suggested that this could be understood in terms of mutual pheromonal interactions among individuals living together or interacting closely. In a successive publication,<sup>11</sup> it was reported the case of a female subject, whose cycle was very regular, being able to lock other women cycles on hers (for a discussion, see also Ref. 8).

Recently, several papers in the physics literature have dealt with the issue of controlling complex networks. In Ref. 12 and 13, for example, a network of identical Kuramoto oscillators is entrained by a single pacemaker, characterized by a different frequency from that of the other oscillators (for a discussion on the effects of the network topology on the synchronization of networks of Kuramoto oscillators, see Refs. 14 and 15).

Because of the distributed nature of complex networks whose dynamics are mainly decentralized, it is feasible to control them by acting locally on part of their nodes and exploiting the coupling effects between these and the rest of the network to achieve the desired goal. In pinning control

schemes,<sup>16–19</sup> some nodes are permanently selected (pinned) to be the network controllers. Specifically, these nodes, referred to as reference sites or pinned sites, play the role of network *leaders/pacemakers*.

The equations for a diffusively coupled complex network under the effect of pinning control, can be generally formulated as follows:

$$\frac{dx_i}{dt} = f(x_i) + \sigma \sum_{j=1}^N \mathcal{L}_{ij} h(x_j) + \sigma \kappa_i B_i (s - x_i), \quad (1)$$

$i=1, \dots, N$ , representing the behavior of  $N$  identical dynamical systems coupled through the network edges.

The first term on the right side of (1) describes the state dynamics of the oscillator at each node,  $\{x_i(t), i=1, \dots, N\}$ , via the nonlinear vector field  $f(x_i)$ ; the second term represents the coupling among pairs of connected oscillators, through a generic output function  $h(x_i)$ , where the coupling gain  $\sigma$  represents the overall strength of the interaction. Information about the weighed network topology is contained in the Laplacian matrix  $\mathcal{L}$ , whose entries  $\mathcal{L}_{ij}$  are zero if node  $i$  is not connected to node  $j \neq i$ , but are negative if there is a direct influence from node  $i$  to node  $j$ , with  $|\mathcal{L}_{ij}|$  giving a measure of the strength of the interaction, and  $\mathcal{L}_{ii} = -\sum_j \mathcal{L}_{ij}$ ,  $i=1, 2, \dots, N$ . In what follows, assume that the network is globally connected, which ensures matrix  $\mathcal{L}$  have only one zero eigenvalue. The control action is directly applied only at the reference nodes, which are indexed by the entries of the binary vector  $B: B_i=1$  ( $B_i=0$ ) if node  $i$  is controlled (not controlled). Hereafter, we assume  $\sum_i B_i \geq 1$ . As commonly assumed in pinning control schemes, such nodes play the role of leading the others toward the desired reference evolution  $s(t)$ .

Here the control input is generated by a simple state-feedback law with respect to the reference evolution  $s(t)$ , which is assumed to satisfy  $\frac{ds}{dt} = f(s)$ , and  $\kappa_i$  is the control gain acting on node  $i$ . Note that, even though there is no reason for considering that the control gains cannot vary among the reference sites, for the sake of simplicity in the rest of the paper, set  $\kappa_i = \kappa, \forall i$ , i.e., assume they are the same at all the reference nodes.

Another problem of non-negligible importance is represented by the choice of the nodes to be pinned from the set of all the network vertices. First of all, one should decide the number of nodes  $m = \sum_i B_i$  to control; in what follows, it is assumed that the number of controlled nodes is ruled by the pinning probability  $p = m/N$ . In many real situations, this decision is often affected by some environmental constraints. In particular, when dealing with biological networks, it becomes particularly evident that both the controlled and the uncontrolled nodes play different but evenly important functions. For example, in the heart, pacemaker and non-pacemaker cells exhibit different phases and amplitudes of their pulsations.

Note that once the number of pinned nodes  $m$  is given, there are  $\binom{N}{m}$  different possibilities of choosing the nodes to control. Usually, the two following strategies for choosing the pinned nodes are considered: (i) Random pinning: The  $m$  pinned nodes are randomly selected with uniform probability

from the set of all the modes. (ii) Selective pinning: The  $m$  pinned nodes are first sorted according to a certain property of the nodes, for instance, the nodes degree or betweenness centrality, then the pinned nodes are chosen in that particular order.

The rest of the paper is outlined as follows. In Sec. II, a definition of controllability is presented for a general complex dynamical network, subject to a decentralized control action. In Sec. III, the effects of heterogeneity in the network degree distribution are discussed. The role of degree correlation is further analyzed in Sec. IV. In Sec. V, linear and square lattices are considered, showing that the distance among the selected reference nodes across the network is an important property to the network controllability. This idea is confirmed by simulations of complex networks with community structure in Sec. VI.

## II. A STRUCTURAL MEASURE OF NETWORK CONTROLLABILITY

Following Ref. 1, here we are interested in the stability of the solutions  $x_1(t)=x_2(t)=\dots=x_N(t)=s(t)$  of the network (1). After standard manipulations,<sup>1,4</sup> this can be evaluated in terms of the dynamics of  $N$  independent blocks in the parameters  $\alpha=\sigma\mu_i$ ,  $i=1, \dots, N$ :<sup>4</sup>

$$\frac{d\eta_i}{dt} = [Jf(s) - \alpha Jh(s)]\eta_i, \quad i = 2, \dots, N, \quad (2)$$

where  $Jf(s)$  and  $Jh(s)$  are the Jacobians of the functions  $f$  and  $h$  calculated about the time varying reference evolution  $s$  and  $\mu_i$ ,  $i=1, \dots, N$  are the eigenvalues of the  $N$ -dimensional structural matrix

$$\mathcal{M} = \{\mathcal{M}_{ij}\} = \begin{pmatrix} \mathcal{L}_{11} + B_{1\kappa_1} & \mathcal{L}_{12} & \dots & \mathcal{L}_{1N} \\ \mathcal{L}_{21} & \mathcal{L}_{22} + B_{2\kappa_2} & \dots & \mathcal{L}_{2N} \\ & \ddots & & \vdots \\ \mathcal{L}_{N1} & \mathcal{L}_{N2} & \dots & \mathcal{L}_{NN} + B_{N\kappa_N} \end{pmatrix}.$$

Hereafter, assume the network to be undirected (and unweighed), which ensures the matrix  $\mathcal{M}$  be symmetric and thus its spectrum be real.<sup>33</sup> Suppose its eigenvalues are sorted as  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_N$ . Note that  $\mathcal{M}$  is not a Laplacian matrix; however, as explained in Ref. 1, the same analysis can be performed in terms of an  $(N+1)$ -dimensional Laplacian matrix, having the same spectrum as  $\mathcal{M}$ , plus one additional zero eigenvalue (for a comparison, the reader is referred to Ref. 1).

Specifically, when all the Lyapunov exponents  $\Lambda(\alpha)$  associated with the  $N$  systems in (2) are negative, the trajectories of all the network systems are found to be stable around the reference evolution  $s(t)$ . Note that differing from the previously reported case of the network synchronizability, this requires the investigation of one more eigenvalue, the one associated with the dynamics along the direction of the synchronization manifold. It is worth noting that theoretically, it is this additional eigenvalue that makes the difference between the synchronizability and the controllability of complex dynamical networks. Moreover, different from the case of the network synchronization, in order to control a network

it is not necessary to have a unique globally connected cluster, while this condition can be replaced by the other one, that in each cluster at least one controller is present (for more details, see Sec. VI).

Now introduce the Master Stability Function (MSF) that associates to each value of the normalized coupling strength  $\alpha$ , the largest Lyapunov exponent  $\Lambda(\alpha)$  of any given system of the form (2). Note that the particular MSF depends only on the choice of the dynamical functions  $f$  and  $h$  but not on the network topology. Moreover, once the MSF is assigned, it is possible to define an interval of values of  $\alpha$ , say  $[\alpha_{\min}, \alpha_{\max}]$ , which corresponds to negative values of the MSF.

In what follows, we will distinguish between the following different situations: (i) when  $\alpha_{\max}$  is finite, the width of the range of values of  $\sigma$ , say  $\Sigma$ , corresponding to a negative MSF, is finite and is an increasing function of the eigenratio  $R = \mu_N / \mu_1$ ; (ii) when  $\alpha_{\max}$  is infinite, the sole  $\mu_1$  gives information about the network controllability: namely, the higher  $\mu_1$  is, the more the network is controllable, i.e., the lower is the critical value of  $\sigma$  above which the reference evolution is stable (note that this fits perfectly the global stability conditions obtained in Ref. 19).

The most important result of this approach is the decoupling of the structural information about the network topology from some particular kind of dynamics at the network nodes. Here, the network structural properties are not only the network topology in terms of the connections and the weights over them, but also the particular choice of the reference sites and the control gains over them (i.e., all the information encoded in the matrix  $\mathcal{M}$ ). This indicates that the network controllability, different from the network synchronizability, can be enhanced by an appropriate choice of the reference sites and of the control gains over them. Therefore, in the rest of the paper, the design of the most effective strategies will be discussed as how to place the controllers over a given network to enhance its controllability.

As a by-product, it becomes possible to compare directly the controllability of different networks topologies in terms of the spectral properties of the matrix  $\mathcal{M}$ . Note that in so doing, one needs to distinguish the two cases (i) and (ii), as indicated above. Specifically, in what follows, the network dynamical systems will be classified respectively as class I or class II according to these two cases.

As explained in Ref. 1, by following this approach it is even possible to compare the controllability and the synchronizability of a given network. In particular, it is possible to compare the widths of the intervals of the coupling gain  $\sigma$  that lead to the stability of the synchronous (or reference) evolutions. However, as shown in Ref. 1, in the case that the number of reference sites  $m \ll N$ , the networks are generally harder to control than to synchronize.

A surprising finding in Ref. 1 is that for class I systems, the network controllability is reduced as the average control gain  $\kappa$  is increased to above a certain value and this property was found over a wide variety of different networks and even lattices.

### III. EFFECTS OF HETEROGENEITY IN THE DEGREE DISTRIBUTION

Heterogeneity in the degree distribution is probably the most important feature that characterizes the structure of real networks. The discovery that the basic structure of many real-world networks is characterized by a power-law degree distribution was first pointed out by Barabasi and Albert in their seminal work,<sup>20</sup> which has been verified by many observations of real networks. Specifically, the analysis of data sets of biological, social, and technological networks has shown that these typically exhibit a power-law degree distributions,  $P(k) \sim k^{-\gamma}$ , which is characterized by high heterogeneity.

In Ref. 5, the Master Stability Function method was used to assess the synchronizability of networks characterized by different degree distributions, and a surprising observation was that the higher the network heterogeneity, the lower their synchronizability. This indicates that the range of values of the coupling strength  $\sigma$ , for which real networks can be synchronized, is particularly narrow with respect to other network architectures. In Ref. 5, this interesting phenomenon was called the *paradox of heterogeneity*.

In this section, we attempt to assess the controllability of a complex network by varying the heterogeneity of its degree distribution. In order to reproduce various networks characterized by different degree distributions, we introduce an appropriate network construction model.

Specifically, we generate a scale-free random network through the static model described in what follows. Firstly, startup with a network of size  $N$ , we assign to each vertex  $i=1, 2, \dots, N$  a weight  $w_i=(i+\theta)^{-\mu}$ , where the Zipf exponent  $\mu$  lies in the range  $[0, 1)$  and  $\theta \ll N$ . Assume that initially no edges are present among the network vertices, then we add edges one by one until  $\mathcal{E}$  connections are created. For each new edge, two vertices are randomly selected, each one with probability proportional to its weight, and they are connected unless a link already exists or the two selected nodes are the same.

By following this construction, the expected degree of node  $i$ , say  $k_i$ , is simply equal to  $k_i=c(i+\theta)^{-\mu}$ , where  $c=2\mathcal{E}/\sum_{j=1}^N(j+\theta)^{-\mu}$ . Thus

$$\begin{aligned} P(k_i \leq \bar{k}) &= P[c(i+\theta)^{-\mu} \leq \bar{k}] \\ &= P\left\{i+\theta > \left[\left(\frac{\bar{k}}{c}\right)^{-1/\mu}\right]\right\} \\ &= \frac{N+\theta - \left(\frac{\bar{k}}{c}\right)^{-1/\mu}}{N-1}. \end{aligned}$$

Then the probability of finding a vertex of degree  $\bar{k}$  is  $P(\bar{k})=c'\bar{k}^{-[1+(1/\mu)]}$ , with  $c'=c^{1/\mu}/[\mu(N-1)]$  and hence the power-law scaling is satisfied with  $\gamma=1+\frac{1}{\mu}$ . Note that the scaling of the degree distribution is independent of  $\theta$ .

The methodology presented above is an extension of the classical *static model* (which has been intensively studied in a number of papers), introduced in Ref. 21; namely, this is recovered in the particular case of  $\theta=0$ .

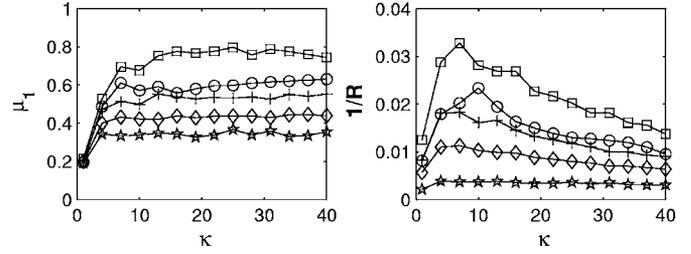


FIG. 1. Networks of  $N=10^3$  nodes,  $\mathcal{E}=3 \times 10^3$  edges,  $\theta=10$ .  $\mu_1$  and  $1/R$  are plotted vs  $\kappa$  in networks characterized by different degree distribution  $P(k) \sim k^{-\gamma}$ . The legend is as follows:  $p=0.05$ ,  $\gamma=2.1$  (stars);  $p=0.05$ ,  $\gamma=3$  (diamonds);  $p=0.05$ ,  $\gamma=4$  (plus);  $p=0.25$ ,  $\gamma=2.1$  (circles);  $p=0.25$ ,  $\gamma=3$  (squares).

The main results are shown in Fig. 1, where  $\mu_1$  and  $1/R$  have been plotted versus the coupling gain  $\kappa$  for networks characterized by different power-law exponents<sup>34</sup> (i.e.,  $\gamma=2.1, 3, 4$ ) and in the cases of a small/large number of controlled nodes (in terms of the probability  $p=0.05, 0.25$ ). Recall that the larger  $1/R$  and  $\mu_1$  are, the more the networks of dynamical systems in class I and class II, are controllable.

Figure 1 clearly shows that by increasing  $\gamma$ , it is possible to enhance the network controllability. This result is particularly surprising, since diffusion dynamics are known to be favored in networks characterized by higher heterogeneity in the degree distribution.<sup>14,22</sup> Moreover, this is in accordance with the phenomenon known as the paradox of heterogeneity in the context of network synchronization.<sup>5</sup> Thus the paradox of heterogeneity affects not only the synchronizability of complex networks, but also their controllability.

Dynamical simulations were carried out involving Rössler oscillators diffusively coupled in the  $x-z$  states (which are known to belong to class I; see also Ref. 1). Namely, we have considered  $N$  identical Rössler oscillators placed at the network vertices; the dynamics at each node  $i$  is described by the following vector field:  $f(x_i)=f(x_{i1}, x_{i2}, x_{i3})=[-x_{i2}-x_{i3}, x_{i1}+0.165x_{i2}, 0.2+(x_{i1}-10)x_{i3}]$ . The output function  $h$  has been chosen, as in Ref. 1, to be  $h(x)=Hx$ , where  $H$  is the matrix,  $[(1 \ 0 \ 0), (0 \ 0 \ 0), (0 \ 0 \ 1)]$ , indicating that the oscillators are coupled through the variables  $x_{i1}$  and  $x_{i3}$  ( $i=1, 2, \dots, N$ ). The asymptotic value of the control error  $E = \frac{1}{(\Delta T)N} \sum_{i=1}^N \int_T^{T+\Delta T} \|x_i(t) - s(t)\| dt$ , with  $\|x\| = |x_1| + |x_2| + |x_3|$ , has been computed under variations of the control gain  $\kappa$  and the number of controlled nodes  $m$ .

The main results are shown in Figs. 1–5, where the theoretical predictions, based on the computation of the eigenratio  $R$ , are shown to be pretty well reproduced by the numerical simulations, involving coupled dynamical systems at the network vertices.

A comparison between random and selective pinning (where the controlled nodes have been chosen in order of decreasing degree) is also shown in Fig. 6. In each subplot networks characterized by different degree distribution exponents are represented [in Fig. 6(d), the network topology is characterized by an exponential decay of the degree distribution, i.e.,  $\gamma=\infty$ ].

Figure 6 deserves detailed comments, as listed below:

- (i) Notice again that (observe the different scales on the y

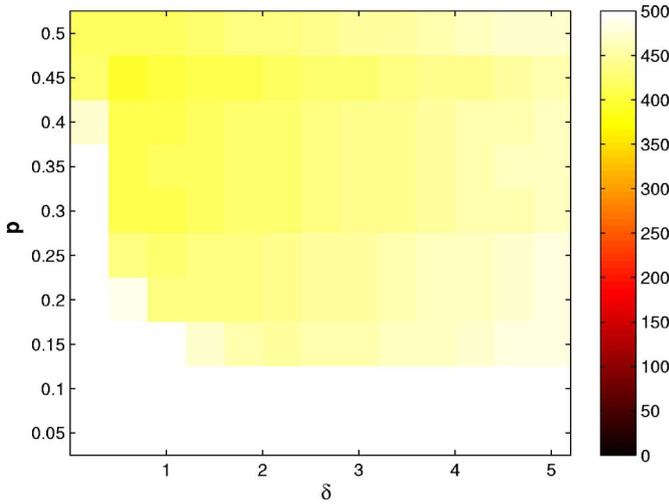


FIG. 2. (Color online) A scale-free network of  $10^3$  nodes with degree distribution exponent  $\gamma=2.1$  is considered. The figure shows the eigenratio  $R$  as varying both the control gain  $\delta=\sigma\kappa$  with ( $\sigma=0.2$ ) and the number of controlled nodes (in terms of the probability  $p$ ).

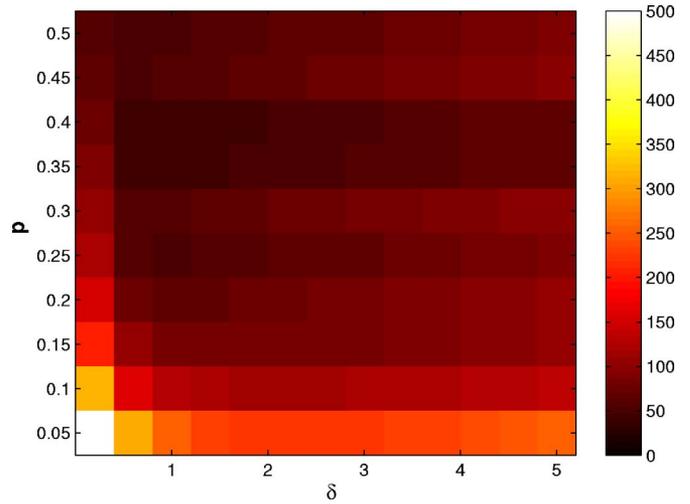


FIG. 4. (Color online) A scale-free network of  $10^3$  nodes with degree distribution exponent  $\gamma=4$  is considered. The figure shows the eigenratio  $R$  as varying both the control gain  $\delta=\sigma\kappa$  with ( $\sigma=0.2$ ) and the number of controlled nodes (in terms of the probability  $p$ ).

axis), networks characterized by higher values of  $\gamma$  are more controllable. The value of the controllability index  $1/R$  is particularly low in the case of highly heterogeneous networks [Fig. 6(a)]. Specifically, in such a case, its increase is shown to saturate for high values of  $m$ , indicating that these networks are particularly hard to control (i.e., further increase in the number of controlled nodes does not lead to further improvements in the network controllability).

- (ii) By comparing the results in each subplot, one can observe that increasing the number of pinned nodes  $m$ , always results in an increased controllability, while by varying  $\kappa$  a more complex behavior emerges. Typically, as  $\kappa$  grows, first an increase and then a decrease of  $1/R$  is observed, indicating the existence of optimal ranges of values of  $\kappa$  in terms of the network controllability (see also Figs. 1–5). These results con-

firm those previously obtained in Ref. 1 for a Barabasi-Albert network characterized by degree distribution,  $P(k) \sim k^{-3}$ .

- (iii) Figure 6 also shows a comparison between random strategies (black lines) and selective strategies (red lines) in choosing the reference sites, where the latter are generally observed to lead to enhanced network controllability. An exception is represented by highly heterogeneous networks, in the case where the number of controlled nodes is sufficiently large. Specifically in such a case, random strategies can be observed to eventually outperform selective strategies [see, e.g., the cases of  $\kappa=60$  and  $30$  in Fig. 6(a)].

#### IV. EFFECTS OF DEGREE CORRELATION

In the preceding section, it was shown that the degree distribution is indeed an important property in affecting the

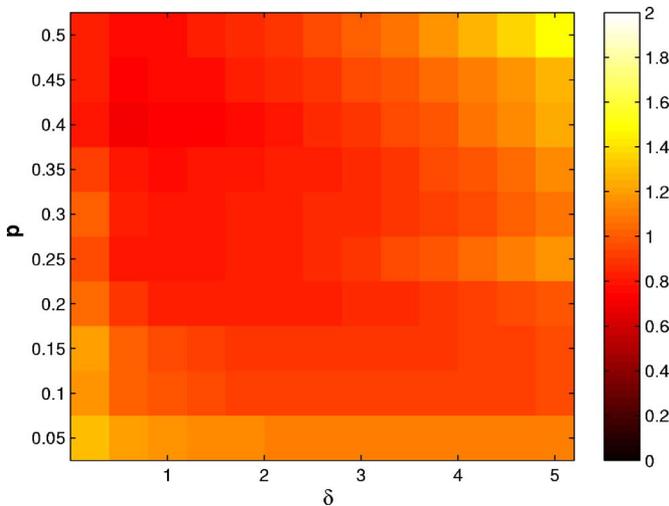


FIG. 3. (Color online) Control error at regime  $E$  as a function of the control gain  $\delta=\sigma\kappa$  with ( $\sigma=0.2$ ) and the probability  $p$  under the same conditions as in Fig. 2.

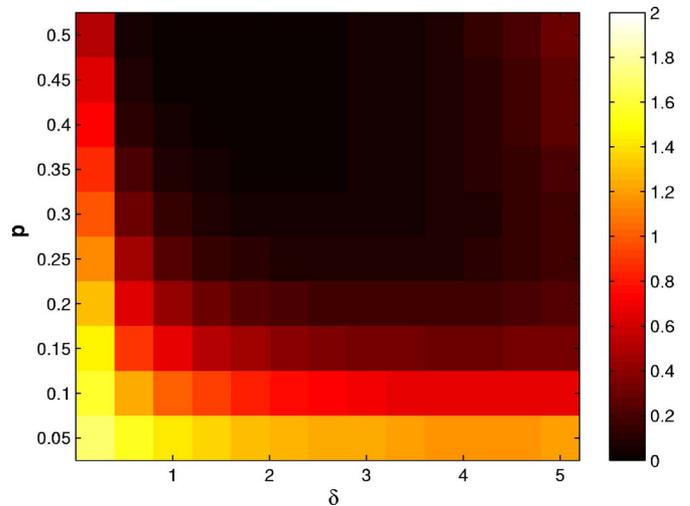


FIG. 5. (Color online) Control error at regime  $E$  as a function of the control gain  $\delta=\sigma\kappa$  with ( $\sigma=0.2$ ) and the probability  $p$  under the same conditions as in Fig. 4.

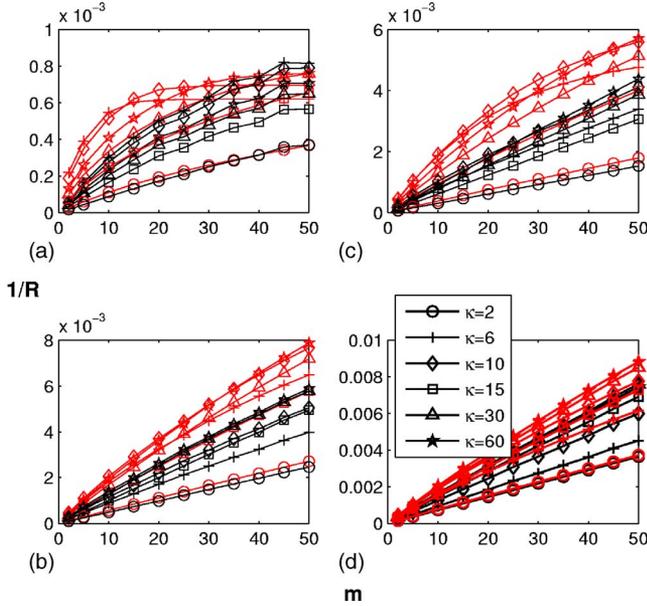


FIG. 6. (Color online) Networks of  $10^3$  nodes. The controllability  $1/R$  is plotted vs the number of controlled nodes  $m$  for networks characterized by different degree distribution exponents:  $\gamma=2.1$  in (a),  $\gamma=3$  in (b),  $\gamma=4$  in (c),  $\gamma=\infty$  in (d). Black (red) is used for random (selective) pinning. Different lines represent different values of the control gain  $\kappa$ , ranging from  $\kappa=2$  to 50.

network controllability. On the other hand, many other distinctive properties have been uncovered to characterize in more detail the structure of real networks, such as, for example, the formation of communities of strongly interconnected nodes, frequently detected in many real networks<sup>23</sup> (for a discussion on the effects of community structure on the network controllability, see Sec. VI), or particular forms of correlation or mixing among the network vertices.<sup>24</sup>

One form of mixing is the correlation among pairs of linked nodes according to some properties at the network nodes. A very simple case is degree correlation,<sup>25</sup> in which vertices choose their neighbors according to their respective degrees. Nontrivial forms of degree correlation have been experimentally detected in many real-world networks, with social networks being typically characterized by assortative mixing (which is the case when vertices are more likely to connect to other vertices with approximately the same degree), and technological and biological networks, by disassortative mixing (which takes place when connections are more frequent between vertices of different degrees). In Ref. 25, this property has been conveniently measured by means of a single normalized index, the Pearson statistic  $r$  defined as follows:

$$r = \frac{1}{\sigma_{q_{k,k'}}^2} \sum k k' (e_{kk'} - q_k q_{k'}), \quad (3)$$

where  $q_k$  is the probability that a randomly chosen edge is connected to a node having degree  $k$ ;  $\sigma_q$  is the standard deviation of the distribution  $q_k$  and  $e_{kk'}$  represents the probability that two vertices at the end points of a generic edge have degrees  $k$  and  $k'$ , respectively. Positive values of  $r$  in-

dicate assortative mixing, while negative values characterize disassortative networks.

The effects of degree correlation on the network synchronizability have been studied in Ref. 26. Specifically in Ref. 26, the network synchronizability has been shown to be enhanced as the network becomes more disassortative (i.e.,  $r$  decreases) for both systems in class I and II (namely, what is observed is that the second smallest eigenvalue varies sensibly with  $r$ , while the largest one is weakly influenced by variations of the network degree correlation). In what follows, we attempt to characterize the effects of degree correlation on the network controllability.

Specifically, by following a strategy similar to the one presented in Refs. 25 and 27, which allows to vary the degree correlation (in terms of variable values of  $r$ ) while keeping the degree distribution fixed, we show that disassortative mixing, i.e., the tendency of high-degree nodes to establish connections with low-degree ones (and *vice versa*), is indeed a desirable network property in terms of its controllability.

The main results are shown in Figs. 7 and 8. Note that in all the cases considered (many simulations were carried out involving different values of the control gain  $\kappa$ , a different number of controlled nodes  $m$ , chosen according to different strategies, and different degree distribution exponents  $\gamma$ ), negative degree correlation has always been found to enhance the network controllability.

Moreover, the effects of degree correlation seem to be strongly emphasized when large gains  $\kappa$  are used to control the network and selective pinning strategies are considered. For example, in the cases represented in Figs. 8(d)–8(f), where selective pinning is used in combination with large control gains at the pinned nodes, the variations in the controllability for networks characterized by different degree-degree mixing, are impressive.

Another interesting phenomenon is observed when different pinning strategies are used to control assortatively mixed networks. In fact, as can be observed by comparing Figs. 7(d)–7(f) and 8(d)–8(f), under certain conditions, random pinning results more effective than selective pinning in controlling such networks.

As we will show in the following, this surprising phenomenon can be explained in terms of the distribution of the controllers over the whole network. Specifically, when the controllers are located at the high degree nodes (selective pinning) and these are all linked together (assortative mixing), this causes a loss of their capability to control the rest of the network. Therefore in such a case, random pinning (i.e., a more uniform distribution of the controllers) turns out to be more effective than selective pinning. On the other hand, when the controllers are located at the high degree nodes and these can maximize their influence being connected to many low degree ones (disassortative mixing), it is observed that the network controllability is strongly enhanced.

## V. LINEAR AND SQUARE LATTICES

As an example of a very simple network, we consider here a linear (monodimensional) lattice of  $N$  nodes. Each node  $i=2, \dots, N-1$  is connected to nodes  $i-1$  and  $i+1$ ; we

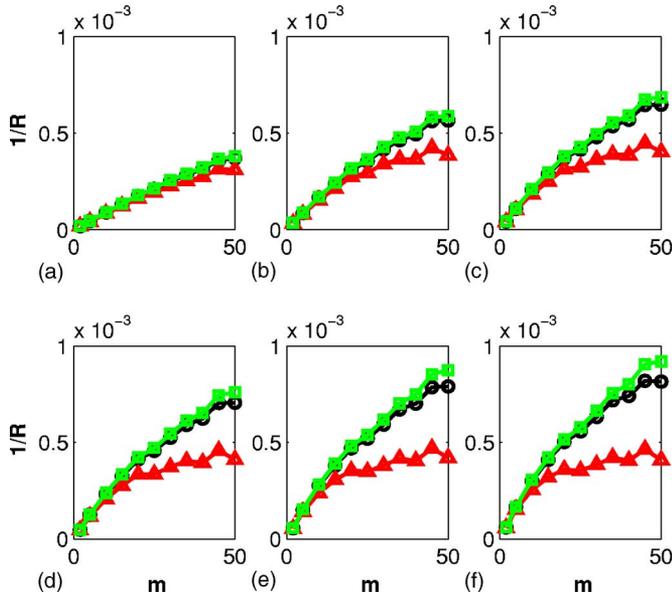


FIG. 7. (Color online) Random pinning of networks characterized by different degree correlation properties,  $N=10^3$ ,  $\mathcal{E}=4 \times 10^3$ ,  $\theta=10$ . Here we consider a scale-free heterogeneous network with degree distribution exponent  $\gamma=2.1$ . The controllability is reported vs the number of controlled nodes  $m$ , by having in each picture a different value of the control gain  $\kappa$ : (a)  $\kappa=2$ , (b)  $\kappa=6$ , (c)  $\kappa=10$ , (d)  $\kappa=15$ , (e)  $\kappa=30$ , (f)  $\kappa=50$ . Different degree correlation properties are indicated by different symbols: Triangles are used in the case of assortative networks ( $r=0.3$ ), squares in the case of disassortative networks ( $r=-0.3$ ), and circles in the case of networks that do not display degree correlation ( $r=0$ ).

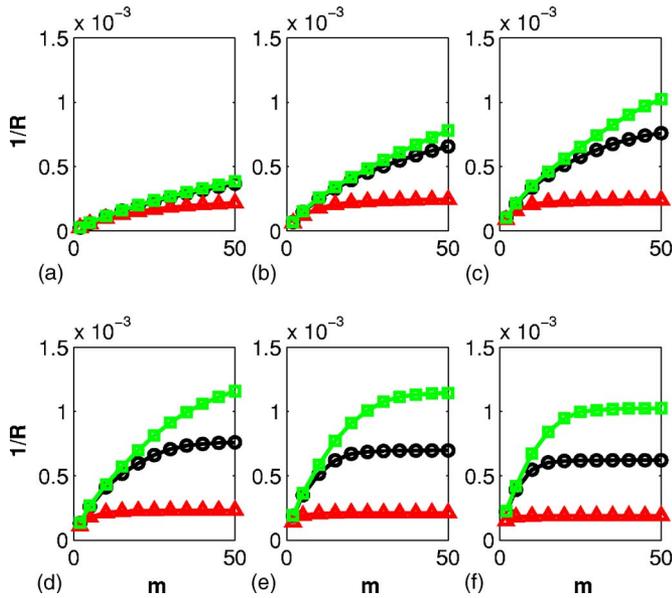


FIG. 8. (Color online) Selective pinning of networks characterized by different degree correlation properties,  $N=10^3$ ,  $\mathcal{E}=4 \times 10^3$ ,  $\theta=10$ . Here we consider a scale-free heterogeneous network with degree distribution exponent  $\gamma=2.1$ . The controllability is reported vs the number of controlled nodes  $m$ , by having in each picture a different value of the control gain  $\kappa$ : (a)  $\kappa=2$ , (b)  $\kappa=6$ , (c)  $\kappa=10$ , (d)  $\kappa=15$ , (e)  $\kappa=30$ , (f)  $\kappa=50$ . Different degree correlation properties are indicated by different symbols: Triangles are used in the case of assortative networks ( $r=0.3$ ), squares in the case of disassortative networks ( $r=-0.3$ ), and circles in the case of networks that do not display degree correlation ( $r=0$ ).

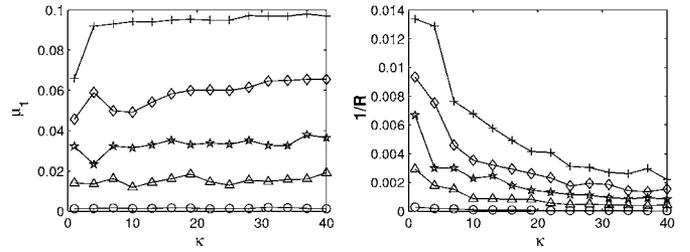


FIG. 9. A monodimensional lattice of  $10^3$  nodes is considered. The controllability indices  $\mu_1$  and  $1/R$  are plotted as functions of the control gain  $\kappa$ . The legend is as follows:  $p=0.05$  (circles),  $p=0.20$  (triangles),  $p=0.30$  (stars),  $p=0.40$  (diamonds), and  $p=0.50$  (plus).

assume periodic boundary conditions, i.e., node 1 is connected to nodes  $N$  and 2, and node  $N$  is connected to nodes  $N-1$  and 1. The controllability of such a network is shown in Fig. 9 as varying both the number of controlled nodes  $m$  and the control gain  $\kappa$ . As Fig. 9 (left-hand plot) shows, using a larger number of controllers is effective in enhancing the network controllability  $\mu_1$ , while only a small improvement is achieved by increasing the control gain  $\kappa$ . On the other hand, increasing the control gain  $\kappa$ , leads to a sharp decrease of the controllability  $1/R$  for the dynamical systems in class I (as shown on the right-hand side plot).

In what follows, we address the issue of which is the best combination of nodes to control this simple model of network, in order to increase its controllability. Imagine one has already selected one node  $i$  at random, say  $i=1$  (without any loss of generality). We now wonder which node  $j$  should be selected then to maximize the network controllability. In Fig. 10, it is shown how the network controllability is affected by the choice of the second node,  $j=2, \dots, N$ . Interestingly, the best choice is performed when the farthest node from  $i$  is selected, i.e.,  $j=501$ . Note that this result is not influenced by the particular value of the control gain  $\kappa$  (as shown in Fig. 10).

Furthermore, we have considered more complex situations in which three or more nodes were to be selected. Interestingly, the combination that optimizes the network controllability is always the one that maximizes the average distance among the selected nodes. Hence, we conjecture that the distance among the selected nodes to be controlled in

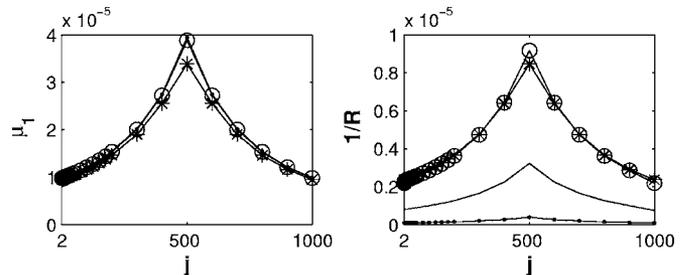


FIG. 10. A monodimensional lattice of  $10^3$  nodes is considered. Assuming that the controlled nodes are node 1 and node  $j$ , the controllability indices  $\mu_1$  and  $1/R$  are plotted vs  $j$ . Different lines indicate different control gains  $\kappa$ :  $\kappa=10^{-1}$  (asterisks),  $\kappa=10^0$  (circles),  $\kappa=10^1$  (no marker), and  $\kappa=10^2$  (dots).

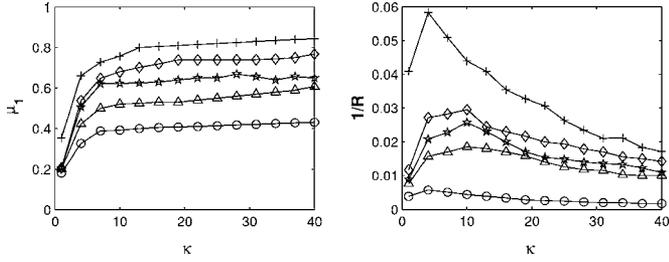


FIG. 11. A periodic square lattice of  $32 \times 32$  nodes ( $z=4$ ) is considered. The controllability indices  $\mu_1$  and  $1/R$  are plotted as functions of the control gain  $\kappa$ . The legend is as follows:  $p=0.05$  (circles),  $p=0.20$  (triangles),  $p=0.30$  (stars),  $p=0.40$  (diamonds), and  $p=0.50$  (plus).

a network is an important aspect that one should consider, when designing pinning control schemes.

As a further example, we consider the case of a square lattice with periodic boundary conditions. Lattices are structured regular networks in which all the nodes have the same degree (also termed coordination number  $z$ ) and are widely used in physics to describe phenomena that take place in ordered extended systems. For example, pinning control of coupled map lattices has been studied in Refs. 16 and 17.

In what follows, we assume the network takes the form of a square lattice, consisting of  $N=l \times l$  nodes, where  $l$  is the side of the lattice. Each node  $i$  can be mapped into a point of integer coordinates  $(x^i, y^i)$ , with  $x^i=1, \dots, l$  and  $y^i=1, \dots, l$ , every site being linked to its  $z=4$  nearest neighbors. We assume periodic boundary conditions and hence the network can be seen as having a toroidal topology in which nodes on the edge of the lattice are connected to those on the opposite edge. This is also known as a Manhattan lattice.

The controllability for such a network is shown in Fig. 11, where both  $\mu_1$  and  $1/R$  are plotted versus the control gain  $\kappa$ . Note that there are intermediate values of the control gain  $\kappa$  which maximize  $1/R$  (as shown in the left panel of Fig. 11). Thus, either too large or too small values of  $\kappa$  can reduce the controllability of square lattices.

Again, as in the preceding case, we are interested in the best strategy to control the lattice, once a given number of controlled nodes (e.g., two) is given. To this aim, a node is selected at random from the network, say  $i$  (whose choice will not affect our results due to the isotropy property of the lattice). Then the other nodes in the network are grouped according to their distance from  $i$ , which ranges between 1 and  $l$ . Note that this time, the choice of which node is selected among all those at a certain distance  $d$  from  $i$  is not ineffective in terms of the network controllability. However, we have checked the variations (in terms of controllability) among all the nodes at a certain distance from  $i$  to be negligible when compared to those among nodes at different distances.

The results are shown in Fig. 12, where both  $\mu_1$  and  $1/R$  are reported, as varying the distance  $d$  between the controlled nodes  $i$  and  $j$  (in the figure, each point has been averaged over different choices of the node  $j$  at a given distance  $d$  from  $i$ ). Again, the best case in terms of controllability is obtained when  $d$  is maximized.

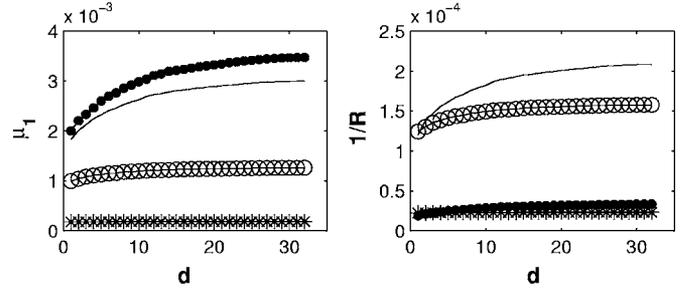


FIG. 12. A square lattice of  $32 \times 32$  nodes is considered. Assuming that two of its nodes have been chosen to be controlled, the controllability indices  $\mu_1$  and  $1/R$  are plotted vs their distance  $d$ . Different lines indicate different control gains  $\kappa$ :  $\kappa=10^{-1}$  (asterisks),  $\kappa=10^0$  (circles),  $\kappa=10^1$  (no marker), and  $\kappa=10^2$  (dots).

This confirms our previous conjecture that the distribution of the controlled nodes over the network is indeed an important property one should consider when designing pinning control schemes. In other words, the more uniformly the nodes are distributed over the network, the higher is their ability to control it.

Previous studies<sup>1,18</sup> have already considered different strategies (i.e., random versus selective strategies) in selecting the best suited reference sites over a given complex network. Therein, it has been proposed that the degree of the selected vertices is indeed an important factor. Here we propose that the uniformity of the distribution of the reference sites over the network (in terms, e.g., of the total distance among them) is an important aspect as well. In the next section, we seek to provide further evidence of this simple general principle, by considering the controllability of complex networks characterized by community structure.

## VI. COMMUNITY STRUCTURE

Synchronizability and synchronization dynamics of networks characterized by community structure have been previously studied in Refs. 28–30. In Ref. 28, the interplay between modular synchronization and global synchronization has been discussed in networks affected by community structure. In Ref. 29, the dynamic time scales of hierarchical synchronization among network communities have been studied, and a connection between the spectral information of the whole Laplacian spectrum and the dynamical process of modular synchronization has been reported.

Here we shall seek to analyze the controllability of networks affected by community structure.<sup>23</sup> Community structure arises when the number of edges between vertices belonging to different communities is considerably smaller than the number of edges falling inside of the communities.<sup>35</sup>

Next we propose a dynamical process that, starting from a generic network configuration and an arbitrary subdivision of its nodes in communities, allows to strengthen the network community structure. Specifically, this is achieved by making the intercommunity connections dynamically rewired inside of the single communities.

Let us start with a given general complex network and assume a simple subdivision of the networks nodes in two communities, say  $C_1$  and  $C_2$ . Say  $N_1$  is the number of nodes

in  $C_1$  and  $N_2 = N - N_1$  is the number of nodes in  $C_2$ . Consider the set  $S$  of all the connections falling between  $C_1$  and  $C_2$  and say  $s$  is its cardinality. At the beginning, the number of links in  $S$  is  $s_0$ . At every single step of the algorithm, the following operations are performed:

- (1) A pair of links is selected from  $S$ . Let us denote by  $v_1$  and  $w_1$  the end points of the two selected edges in  $C_1$ , and  $v_2$  and  $w_2$  their end points in  $C_2$ .
- (2) In the case in which  $v_1$  and  $w_1$  (and  $v_2$  and  $w_2$ ) are not already connected, the selected links are eliminated from the network and two new connections, namely  $(v_1, w_1)$  and  $(v_2, w_2)$ , are created.
- (3) In the case in which the switch in (2) is successfully performed,  $s$  is decreased by 2.

Note that the algorithm described above is ergodic over the degree distribution.<sup>24</sup>

Hereafter, for the sake of simplicity, we will consider a network generated by the algorithm described in Sec. III, characterized by exponential decay of the degree distribution (i.e.,  $\gamma = \infty$ ). One-fourth of the nodes are then randomly selected to form the first community  $C_1$ . The remaining ones are those in  $C_2$ . We set the pinning probability to  $p = 0.1$  and randomly choose the reference sites among all the networks nodes.

In Figs. 13(a) and 13(c), the behavior of the network controllability is reported as  $s/s_0$  is varied according to the process described above (note that in so doing, only the connections among the network nodes are rewired, without modifying the choice of the reference sites).

The figures show high sensitivity of the controllability with respect to the choice of the control gain  $\kappa$ , ranging between  $10^{-3}$  and  $10^2$ . However, neither  $\mu_1$  nor  $1/R$  is influenced by the dynamical links redirection process proposed in this section. Even in the limit case in which  $s = 0$ , which corresponds to the network being disconnected in two separate clusters, the network controllability is not observed to go to 0 (however, this is true as long as at least one controller is present in each of the isolated clusters).

It is worth noting that this represents another main difference between control and synchronization of complex networks. Specifically, in order to control a network, it is not necessary to have a unique globally connected cluster. Namely, this condition can be replaced by the one, that states that in each isolated cluster, at least one controller is present.

So far, we have assumed the reference sites to be uniformly randomly selected from the set of all the network nodes (i.e., both  $C_1$  and  $C_2$ ). A different situation is instead shown in the right panels of Figs. 13, where it is assumed that about 95% of the reference sites belong to  $C_1$  and only the remaining 5% belong to  $C_2$ . Similarly to Sec. V, we do this with the aim of unveiling how the distribution of the reference sites among the network nodes affects its controllability (but here, different from Sec. V, we consider complex random networks). In so doing, we compare the controllability of modular networks according to the distribution of the controllers among their different communities.

Furthermore, this is motivated by real networks observation. For instance, the control of the synchronous beat of the

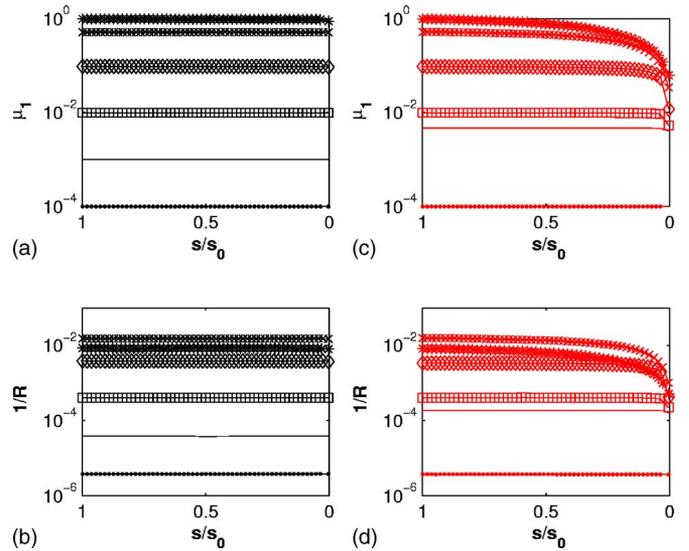


FIG. 13. (Color online) A random network of  $5 \times 10^2$  nodes composed of two communities is considered. The first one,  $C_1$ , includes 125 nodes and the second one,  $C_2$ , includes the remaining (375) ones. The network controllability  $\mu_1(1/R)$  is plotted vs  $s/s_0$  in (a) and (b) [(c) and (d)] as  $s$  is varied between 1 and 0, according to the technique described in the text. The pinning probability is assumed to be  $p = 0.1$ . Different lines indicate several values of the control gain  $\kappa$ :  $\kappa = 10^{-3}$  (points),  $\kappa = 10^{-2}$  (no marker),  $\kappa = 10^{-1}$  (squares),  $\kappa = 10^0$  (diamonds),  $\kappa = 10^1$  (times), and  $\kappa = 10^2$  (asterisks). The left plots (a) and (c) show the cases in which the controllers are evenly distributed among the two communities. The right plots (b) and (d) show a situation in which 95% of the controllers belong to  $C_1$  and the rest to  $C_2$ . Observe the logarithmic scale on the y axis.

heart cells is exerted by the pacemaker cells, which are grouped together at the sinoatrial node.

The simulations in the right panels of Figs. 13 clearly show that in this case, as  $s$  is dynamically reduced from 1 to 0, both  $\mu_1$  and  $1/R$  are influenced by the emergence of the network community structure. In particular, the nonuniform distribution of the reference sites over the network reduces the network controllability.

Loosely speaking, this is because the number of reference sites in one of the communities (i.e.,  $S_2$ ) is too low, so that this part of the network is *out of control*. In other words, as it is clearly shown in Figs. 13(a)–13(d), the emergence of clusters of reference sites is not a desirable property in terms of the network controllability.

In the case of the heartbeat control, we conjecture that the cluster organization of the pacemakers might be explained in terms of the self-sustained oscillations generated by the interactions among the pacemakers themselves. Thus in such a case, one should consider (i) the influence of the reference sites on the other nodes, (ii) the dynamics among the noncontrolled nodes, and also (iii) the dynamics occurring among the pacemakers as well. Note that in the case of the heartbeat, different types of dynamics characterize the reference and the other sites, as witnessed by the difference in the phase and the amplitudes of their pulsations. This represents an interesting subject for future research activity.

## VII. CONCLUSIONS

Motivated by the results reported in a recent paper, where a technique was introduced to describe the controllability

bility of networks under pinning control schemes, in this paper we have compared several network topologies in terms of their controllability. Specifically, we have considered networks in which two different layers of dynamical nodes co-exist: the uncontrolled sites and the reference (controlled) ones, where the latter play the role of leading the whole network to evolve toward a desired reference evolution.

In Ref. 1, a surprising property was pointed out (for the dynamical systems in class I), i.e., the network controllability is reduced when the average control gain  $\kappa$  is increased to beyond a certain value. Interestingly, this property has been confirmed here to be reproduced over a wide variety of different network topologies and even lattices.

This finding is likely to offer an explanation to the daily-life experience that in order to deal with complex systems/situations, one should adopt the right dose of strength in the control action: either a too weak or too strong action could lead to a loss of control over the whole system/situation.

In this paper, we have considered the effects of heterogeneity in the degree distribution, degree correlation, as well as community structure on the network controllability. High heterogeneity in the degree distribution has been found to reduce the network controllability, while disassortative mixing enhances it. Moreover, motivated by the observations on some simple networks, such as linear and square lattices, we have conjectured that a uniform distribution of the reference sites over a network is indeed important regarding its controllability. This finding could be useful for future control theory of complex networks.

This has been confirmed also when more complex topologies have been considered. In particular, in the case of assortatively mixed networks (characterized by positive degree correlation), we have found that controlling the group of the high degree nodes (which is commonly believed to enhance the control performance) is indeed an ineffective strategy. At the same time, in the case of networks characterized by strong community structure, it has been shown that placing most of the controllers within the same community can lead to a loss of control over the whole network.

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- <sup>33</sup>We wish to emphasize that the assumption of undirected and unweighed networks is made here for the sake of simplicity and without loss of generality. The MSF approach, presented in this paper, is also valid under very general assumptions about the network topology [by allowing eventually the spectrum of the matrix  $\mathcal{M}$  to be complex (Ref. 6) and even the matrix  $\mathcal{M}$  to be nondiagonalizable (Ref. 31)].
- <sup>34</sup>In particular, in the case in which the network turned out to be not globally connected (i.e., some isolated nodes or clusters of few nodes connected together appeared), the spectral properties of the giant component have been considered.
- <sup>35</sup>Actually it would be more correct to affirm that community structure is detected when the number of connections falling between different communities is considerably lower than one would expect by considering another network characterized by the same degree distribution but completely random with respect to any other aspect (Ref. 32).

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