Quantum circuits of T-depth one

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We give a Clifford + T representation of the Toffoli gate of T-depth one, using four ancillas. More generally, we describe a class of circuits whose T-depth can be reduced to one by using sufficiently many ancillas. We show that the cost of adding an additional control to any controlled gate is at most eight additional T gates and T-depth two. We also show that the circuit THT does not possess a T-depth one representation with an arbitrary number of ancillas initialized to $|0\rangle$.

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I. INTRODUCTION

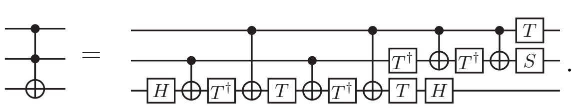
It is known that the gates of the Clifford group, together with the single-qubit non-Clifford gate

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix},$$

form a good universal gate set for fault-tolerant quantum computation [1]. The decomposition of arbitrary gates into this Clifford + T set, either exactly or to within some given accuracy ϵ, is an important problem [2]. It is often desirable to find decompositions that are optimal with respect to a given cost function. The exact cost function used is application dependent; some possibilities are the total number of gates, the total number of T gates, the circuit depth, and/or the number of ancillas used.

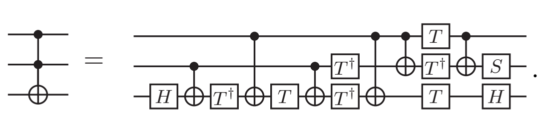
Amy *et al.* [3] recently proposed T-*depth* as a cost function. The idea is to count the number of T *stages* in a circuit, rather than the number of T gates. A T stage is a group of one or more T and/or $T^\dagger$ gates on distinct qubits that can be performed simultaneously. Note that, for the purpose of computing T-count or T-depth, the gates T and $T^\dagger$ can be treated interchangeably, due to the identity $T^\dagger = TS^\dagger$.

To illustrate the concept of T-depth, consider the standard decomposition of the Toffoli gate into the Clifford + T set, as given in Ref. [4]:



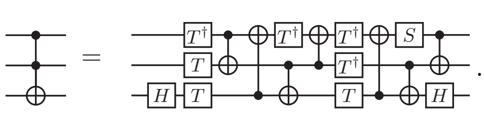
(1)

This decomposition has T-count seven, and in the exact form written, it has T-depth six, because the fourth and fifth T gates form a single T stage. Using trivial commutations, the circuit (1) can easily be reduced to T-depth four:



(2)

Amy *et al.* [3] further improved the T-depth of the Toffoli gate to three, using the following circuit. They conjecture that for circuits without ancillas, this T-depth is optimal:



(3)

The purpose of this paper is to show that, with the use of ancillas, the T-depth of the Toffoli gate, and of many (but not all) other circuits, can be reduced to one. This may be useful in quantum computing architectures where T gates are expensive and ancillas are cheap.

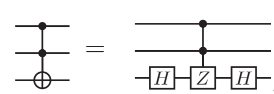
II. A T-DEPTH ONE REPRESENTATION OF THE TOFFOLI GATE

Recall that the Clifford group for any number of qubits is generated by the Hadamard gate H, the phase gate $S = T^2$, the controlled-NOT gate, and unit scalars. As usual, we write X, Y, and Z for the Pauli operators:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Toffoli gate is a doubly controlled NOT gate. It is equivalent to a doubly controlled Z gate via a basis change:



(4)

Now consider a computational basis state $|xyz\rangle$, where $x,y,z \in \{0,1\}$. The effect of the doubly controlled Z gate is to map $|xyz\rangle$ to $(-1)^{xyz}|xyz\rangle$. Let us write "$\oplus$" for modulo-2 addition in $\{0,1\}$, and "$+$" and "$-$" for the usual addition and subtraction of integers. We then have the following inclusion-exclusion style formula for $x,y,z \in \{0,1\}$:

$$\begin{aligned} 4xyz = x + y + z - (x \oplus y) - (y \oplus z) \\ - (x \oplus z) + (x \oplus y \oplus z). \end{aligned} \tag{5}$$

This is easy to prove by case distinction, or algebraically using $x \oplus y = x + y - 2xy$. Now let $\omega = (-1)^{1/4} = e^{i\pi/4}$. From (5), we have

$$(-1)^{xyz} = \omega^{4xyz} = \omega^x \omega^y \omega^z (\omega^\dagger)^{x\oplus y} (\omega^\dagger)^{y\oplus z} (\omega^\dagger)^{x\oplus z} \omega^{x\oplus y\oplus z}. \tag{6}$$

Note that $T|x\rangle = \omega^x|x\rangle$, and therefore, the doubly controlled Z gate can be implemented by applying T gates to qubits in states $|x\rangle$, $|y\rangle$, $|z\rangle$, and $|x \oplus y \oplus z\rangle$, and $T^\dagger$ gates to qubits in states $|x \oplus y\rangle$, $|y \oplus z\rangle$, and $|x \oplus z\rangle$. This can be done in any order, or even in parallel, using four ancillas, as shown in

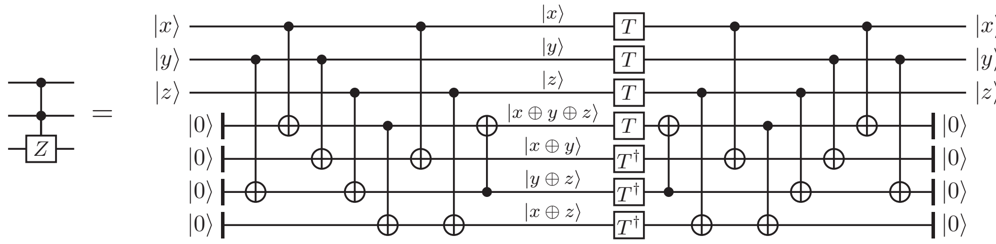


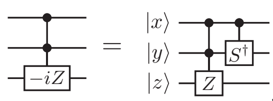
FIG. 1. T-depth one representation of the Toffoli gate.

Fig. 1. Combining this with Eq. (4), we obtain a representation of the Toffoli gate of T-depth one and overall depth seven.

Remark 2.1. It is interesting to note that the decompositions of Nielsen and Chuang (1) and Amy *et al.* (3) follow precisely the same pattern; i.e., they can both be seen to be direct implementations of Eq. (6). The only difference is that in each of the circuits, one of the T gates has been needlessly decomposed into $T^\dagger$ and S.

III. AN APPLICATION TO MULTIPLY CONTROLLED GATES

Consider a doubly controlled $(-iZ)$ gate:

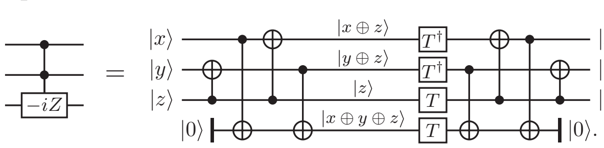


(7)

The doubly controlled Z gate is a diagonal gate whose effect is given by Eq. (6). The controlled-$S^\dagger$ gate is a diagonal gate whose effect is given by $(-i)^{xy} = (\omega^\dagger)^x(\omega^\dagger)^y\omega^{x\oplus y}$. It follows that the combined effect of the two gates is

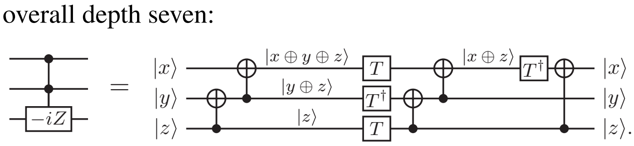
$$(-1)^{xyz}(-i)^{xy} = \omega^z(\omega^\dagger)^{y\oplus z}(\omega^\dagger)^{x\oplus z}\omega^{x\oplus y\oplus z}, \tag{8}$$

which therefore requires a T-count of only four. Using one ancilla, this can be achieved with T-depth one and overall depth five:



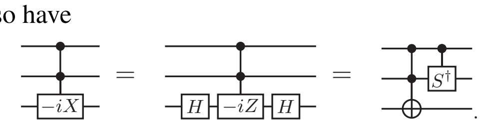
(9)

Alternatively, one can find an implementation that uses no ancilla. It uses fewer overall gates, but has T-depth two and overall depth seven:



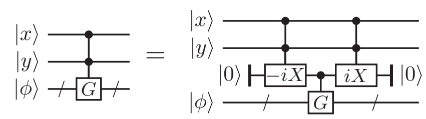
(10)

We also have



(11)

Suppose we have a Clifford + T representation of some controlled quantum gate G, and we wish to obtain an efficient Clifford + T representation of a doubly controlled G gate. Using (9), (11), and (12), the cost of doing so is at most eight additional T gates, increasing the T-depth by at most 2, and the overall depth by at most 14, using two ancillas:

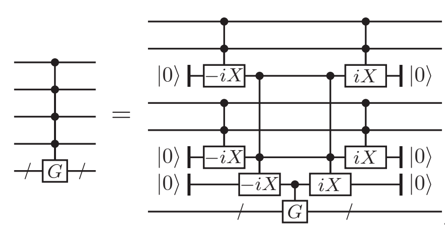


(12)

Note that the cost of the additional control, in terms of the overall gate count, is 28 [2 times 12 gates from Eq. (9) and 2 times 2 Hadamard gates from Eq. (11)]. This can be reduced to 26 by leaving the ancilla in Eq. (9) in state $|x\rangle$ instead of $|0\rangle$; however, doing so requires carrying this ancilla during the computation of G, which may involve a tradeoff.

If (10) is used instead of Eq. (9), the overall gate count cost of Eq. (12) decreases to 22, and the ancilla use to one. However, the depth and T-depth cost increase to 18 and 4, respectively.

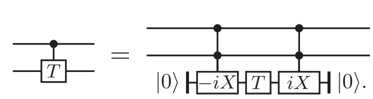
Remark 3.1. The above construction can be iterated to add n additional controls to a controlled gate at the cost of T-count $8n$ and T-depth $2\lfloor\log_2 n + 1\rfloor$. The logarithm in the expression for T-depth arises because a pair of T stages is sufficient to *double* the number of controls, as shown here for $n = 3$:



(13)

For example, this yields an implementation of a triply controlled NOT gate with T-count 15 and T-depth three (7 T gates for the Toffoli gate, and 8 T gates for the additional control); or a quintuply controlled NOT gate with T-count 31 and T-depth five. It is not currently known whether any of these T-counts or depths are optimal.

Remark 3.2. Because the T gate is diagonal with $T|0\rangle = |0\rangle$, it can be regarded as a controlled gate, namely, a controlled global phase change. Therefore, we can use the above procedure to implement a controlled-T gate with T-count nine as follows:



(14)

Using (9), we obtain T-depth 3, depth 15, and gate count 29 with two ancillas. As before, by leaving the ancilla of Eq. (9) in state $|x\rangle$ instead of state $|0\rangle$, the gate count can be reduced to 27. Alternatively, using (10), we obtain T-depth 5, depth 19, and gate count 27 with one ancilla. Except for slightly improved overall gate counts, these results are the same as those in Ref. [3].

IV. T-DEPTH ONE REPRESENTATION OF ALMOST CLASSICAL CIRCUITS

It is straightforward to generalize the construction of Sec. II to circuits built up from T and *almost classical* gates.

Definition 4.1. A unitary operator is *classical* if it is given by a permutation of computational basis states and *diagonal* if its matrix representation is diagonal in the computational basis. Let us call an operator *almost classical* if it can be written as a product of a classical operator and a diagonal operator.

The almost classical operators obviously form a group. Of the 24 single-qubit Clifford operators (taken modulo global phase), exactly 8 are almost classical; they form the subgroup generated by S and X.

Definition 4.2. Let $\mathcal{C}$ be a set of gates. We say that a circuit is *$\mathcal{C}+T$-representable* if it can be built with gates from $\mathcal{C}\cup\{T\}$ and their inverses. We say that such a circuit has *T-depth n* (*relative to $\mathcal{C}$*) if it can be written using only gates from $\mathcal{C}$ and n T stages.

Theorem 4.1. Let $\mathcal{C}$ be any set of almost classical gates, containing the controlled-NOT gate. Using ancillas, any $\mathcal{C}+T$-representable n-qubit circuit can be written of T-depth one (relative to $\mathcal{C}$).

Proof. The proof idea is simple. Each T gate in the circuit is a $\pi/4$ phase change conditioned on some Boolean combination of the inputs. Intuitively, one may copy each such Boolean condition to an ancilla, execute all T gates in parallel, uncompute the ancillas, and finally recompute the output.

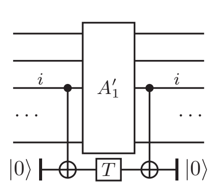
The formal proof proceeds by induction on circuits. For each $\mathcal{C}+T$-representable n-qubit circuit A, we use induction to construct $\mathcal{C}+T$-representable circuits A_1 and A_2 such that A_1 is diagonal and has T-depth at most one, A_2 has T-depth 0, and $A = A_2 \circ A_1$.

The base case occurs when $A = I$ is the identity circuit. In this case, we can let $A_1 = A_2 = I$, and there is nothing to show.

For the induction step, suppose A is of the form $A' \circ G$, where G is a single gate. By induction hypothesis, there is a decomposition $A' = A_2' \circ A_1'$ satisfying the above conditions.

(i) Case 1: G is not equal to T or $T^\dagger$. In this case, we let $A_1 = G^\dagger \circ A_1' \circ G$ and $A_2 = A_2' \circ G$. Then trivially, $A = A_2 \circ A_1$, and A_1 and A_2 have the required T-depths. Moreover, since G is almost classical, A_1 is diagonal.

(ii) Case 2: G is T, applied to the ith qubit. In this case, we let

$A_1 =$  (15)

and $A_2 = A_2'$. Since A_1' is diagonal, so is A_1, and it follows that the ancilla is uncomputed correctly. Moreover, A_1 is equivalent to $A_1' \circ G$, and therefore, $A = A_2 \circ A_1$. Finally, since A_1' has T-depth of at most one, so does A_1.

(iii) Case 3: G is $T^\dagger$, applied to the ith qubit. This is entirely analogous to case 2. ■

A similar result appears in Sec. 6.4 of version 2 of Ref. [3], but with a proof that is quite different.

Note that the gate set $\mathcal{C}$ in Theorem 4.1 is not necessarily assumed to consist of Clifford gates. For example, if on some hypothetical architecture, T gates are expensive but Toffoli gates are cheap, one can include the Toffoli gate in the set $\mathcal{C}$.

In general, the proof of Theorem 4.1 increases the size of the circuit, but only by a constant factor. In practice, it is often possible to find a much smaller circuit than the one constructed in the proof.

If we take $\mathcal{C} = \{S, X, \text{CNOT}\}$ and apply Theorem 4.1 to circuit (1) (excluding the initial and final Hadamard gate), we obtain another T-depth one representation of the Toffoli gate.

We also note that there is a trade-off between T-depth and the number of ancillas. The procedure of the proof of Theorem 4.1 adds one ancilla for each T gate. However, by splitting a circuit with T-count n into two circuits with T-count $\lceil n/2 \rceil$ each, it is clear that one can approximately half the number of ancillas by doubling the T-depth and so forth.

V. SOME CIRCUITS CANNOT BE WRITTEN WITH T-DEPTH ONE

The result of the previous section shows that any two T stages can be combined into a single T stage, provided that they are only separated by almost classical gates. One may wonder whether perhaps *all* Clifford + T circuits can be written of T-depth one, using a sufficient number of ancillas initialized to $|0\rangle$. We show that this cannot be done.

Theorem 5.1. The single-qubit operator THT cannot be implemented as a Clifford + T circuit of T-depth one, using an arbitrary number of ancillas initialized to $|0\rangle$. This is true even if the ancillas are not required to be returned to their initial state at the end of the computation.

Before proving the theorem, we start with a general observation about Clifford + T circuits of T-depth one.

Proposition 5.1. Let U be an n-qubit Clifford + T circuit of T-depth one. Let $|\phi\rangle$ be any single-qubit state, and consider

$$|\psi\rangle = U(|\phi\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle).$$

Consider the $\{+1,-1\}$-valued Pauli observable X applied to the first qubit of ψ; denote its expected value by $E_{|\phi\rangle}$. Suppose $E_{|+\rangle}$ is nonzero. Then

$$\frac{E_{|0\rangle}}{E_{|+\rangle}}$$

is a rational number.

Proof. The expected value of the observable X on the first qubit of $|\psi\rangle$ is

$$\begin{aligned} E_{|\phi\rangle} &= \langle\psi|\,(X \otimes I \otimes \cdots \otimes I)\,|\psi\rangle \\ &= \langle\phi,0,\ldots,0|\;U^\dagger(X \otimes I \otimes \cdots \otimes I)U\;|\phi,0,\ldots,0\rangle. \end{aligned} \tag{16}$$

We analyze the structure of $U^\dagger(X \otimes I \otimes \cdots \otimes I)U$. Since U is of T-depth one, it can be written as $U = U_1 \circ U_2 \circ U_3$, where U_1 and U_3 are Clifford circuits and $U_2 = T \otimes \cdots \otimes T \otimes I \otimes \cdots \otimes I$. Since U_1 is Clifford, we know that $U_1^\dagger(X \otimes I \otimes \cdots \otimes I)U_1$ is a Pauli operator

$$U_1^\dagger(X \otimes I \otimes \cdots \otimes I)U_1 = \pm A_1 \otimes \cdots \otimes A_n, \tag{17}$$

where each $A_i \in \{X,Y,Z,I\}$. Using the relations

$$T^\dagger I T = I, \quad T^\dagger Z T = Z,$$
$$T^\dagger X T = \frac{1}{\sqrt{2}}X - \frac{1}{\sqrt{2}}Y, \quad T^\dagger Y T = \frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y,$$

we find that

$$\begin{aligned} U_2^\dagger(\pm A_1 \otimes \cdots \otimes A_n)U_2 &\\ = \pm(T^\dagger A_1 T) \otimes \cdots \otimes \left(T^\dagger A_{n_1} T\right) \otimes A_{n_1+1} \otimes \cdots \otimes A_n &\\ = \lambda P_1 + \lambda P_2 + \cdots + \lambda P_m, & \end{aligned} \tag{18}$$

where each P_j is an n-qubit Pauli operator. The key observation here is that the *same* factor λ occurs in front of each (possibly signed) summand, and λ is independent of $|\phi\rangle$. In fact, we have $\lambda = (\frac{1}{\sqrt{2}})^k$, where k is the number of times the operators X and Y occur among $A_1, \ldots, A_{n_1}$. Let

$$Q_j = U_3^\dagger \; P_j \; U_3. \tag{19}$$

Since U_3 is Clifford, this is again some Pauli operator, say

$$Q_j = (-1)^{q_j} B_{j,1} \otimes \cdots \otimes B_{j,n}. \tag{20}$$

Combining (17) through (20), we find

$$\begin{aligned} U^\dagger(X \otimes I \otimes \cdots \otimes I)U &= \lambda Q_1 + \lambda Q_2 + \cdots + \lambda Q_m \\ &= \lambda \sum_{j=1}^{m} (-1)^{q_j} B_{j,1} \otimes \cdots \otimes B_{j,n}. \end{aligned} \tag{21}$$

Combining this with Eq. (16), we get

$$E_{|\phi\rangle} = \lambda \sum_{j=1}^{m} (-1)^{q_j} \langle\phi|B_{j,1}|\phi\rangle \; \langle 0|B_{j,2}|0\rangle \cdots \langle 0|B_{j,n}|0\rangle. \tag{22}$$

Since each $B_{j,i} \in \{X,Y,Z,I\}$ is a Pauli operator, it follows that $E_{|\phi\rangle}/\lambda$ is rational (indeed, an integer) for $|\phi\rangle \in \{|0\rangle,|+\rangle\}$. The claim then immediately follows. ■

Proof of Theorem 5.1. For $U = THT$, we compute

$$U^\dagger X U = \frac{1}{2}X + \frac{1}{2}Y + \frac{1}{\sqrt{2}}Z,$$

and therefore

$$E_{|0\rangle} = \langle 0|U^\dagger X U|0\rangle = \frac{1}{\sqrt{2}}$$

and

$$E_{|+\rangle} = \langle +|U^\dagger X U|+\rangle = \tfrac{1}{2}.$$

Since $E_{|0\rangle}/E_{|+\rangle}$ is irrational, the claim immediately follows from Proposition 5.1. ■

VI. CONCLUSION

We found a class of circuits whose T-depth can be reduced to one, by using a sufficient number of ancillas. We also showed that there are circuits whose T-depth cannot be reduced to one, regardless of the number of ancillas used. It remains an open problem how to determine the minimal T-depth or T-count of any given Clifford + T circuit.

ACKNOWLEDGMENTS

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